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A CHARACTERIZATION OF ABSOLUTE RETRACTS OF n-CHROMATIC GRAPHS

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A recursive characterization of the absolute retracts in the class of n-chromatic (connected) graphs is given.

1. Introduction

Retracts of graphs have been considered by several authors (see the references); in particular, Hell [2] has given a description of the absolute retracts of bipartite graphs. In this note, we want to continue these studies and give (for arbitrary $n \geq 2$) a recursive characterization of the absolute retracts of n-chromatic graphs.

Some notation is needed first.

All graphs occurring in this paper are assumed to be finite and connected, without loops or multiple edges. $V(G)$ ($E(G)$) denotes the set of vertices (edges) of a graph G, the edge connecting vertices a and b is denoted by ab. A *homomorphism* $f: G \to H$ is a mapping from $V(G)$ to $V(H)$ preserving all edges of G; this implies that connected vertices are not identified by a homomorphism. H is a *subgraph of* G if $V(H) \subseteq V(G)$, and H has all edges inherited from G. Finally, H is a *retract* of G if H is a subgraph of G, and there is a homomorphism $f: G \to H$ with $f(h) = h$ for all $h \in V(H)$; in this case, f is called *retraction.*

Definition 1.1. Let $n \geq 2$ be a natural number. By AR_n, we denote the class of all absolute retracts of n-chromatic graphs, i.e., $G \in AR_n$ if and only if
- G is n-chromatic, and
- whenever G is an isometric subgraph of G' which also n-chromatic, then there is a retraction of G' onto G.

By *n-chromatic,* as usual, we mean there is a homomorphism onto K_n, the complete graph on n vertices, and n (which is also called the *chromatic number* of G) is the smallest such natural number. It is easy to see that retractions preserve the chromatic number.

A subgraph H of G is called *isometric* if distances between vertices are the same in H as in G: $d_H(h, h') = d_G(h, h')$ for all $h, h' \in V(H)$. The condition

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D_4 D_5

Fig. 1

'isometric' in the above definition comes from the easy observation that a retract is always isometric.

We want to conclude the introduction with some examples and some remarks on absolute retracts.

Trivially, $K_n \in AR_n$.

It is less trivial to see that $D_4 \in AR_3$, but $D_5 \notin AR_3$ (see Fig. 1). This will follow from our Main Theorem.

Another element of AR_3 is G_9 shown in Fig. 2.

Lemma 1.2. *If $G \in AR_n$ and $v \in V(G)$, then there is a subgraph of G isomorphic to K_n which contains v.*

Proof. Simply adjoin new vertices $x_1, \ldots, x_{n-1}$ to G pairwise connected and also neighbours of v, that means $v, x_1, \ldots, x_{n-1}$ form a subgraph isomorphic to K_n of the new graph G'. Since G is isometric in G' which also n-chromatic, there is a retraction of G' onto G which produces the desired copy of K_n inside G. □

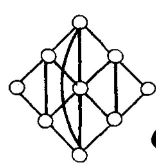
The graph D_5 in Fig. 1 shows the condition of the lemma is not sufficient to characterize AR_n. Similar techniques as in the above proof are used throughout the paper. We only remark that nearly the same proof can be used to show: if $G \in AR_n$, and H is a subgraph of G isomorphic to K_m $(m < n)$, then there is a subgraph of G isomorphic to K_n which contains H.

2. Two important propositions

For the following two results, cf. also Hell [1].

Proposition 2.1. *Let $G \in AR_n$, and let H be an isometric subgraph of G. Then H is a retract of G if and only if $H \in AR_n$.*

Proof. For the non-trivial direction, let $r: G \to H$ be a retraction, and let H be an isometric subgraph of some n-chromatic H'. To avoid set-theoretic confusion,



G_9

Fig. 2

we may assume that $V(G) \cap V(H') = V(H)$. Let G' be the union of G and H' (which is meant in the obvious sense). Obviously, G is isometric in G', and G' is n-chromatic. But now, from $G \in AR_n$, we get a retraction $r': G' \to G$, and $r \circ r'/H'$ is the desired retraction from H' onto H. □

The next proposition can be viewed as the base of the recursion which will be our main result. As usual, the diameter of G, i.e., the maximal distance occurring between vertices of G, is denoted by diam G.

Proposition 2.2. *Let G be n-chromatic. The following are equivalent:*

(i) *For each colouring $c: G \to K_n$ and for each $i \in V(K_n)$, there is a $z_i \in V(G)$ with $z_i v \in E(G)$ for all $v \in V(G)$ with $c(v) \neq i$;*

(ii) *$G \in AR_n$, and* $\operatorname{diam} G \leq 3$ *if* $n = 2$, $\operatorname{diam} G \leq 2$ *if* $n \geq 3$.

Proof.

(i) → (ii): Let G be an isometric subgraph of G' with colouring $c': G' \to K_n$. The map $r: G' \to G$ defined by

$$r(x) := \begin{cases} z_i & \text{if } x \notin V(G) \text{ and } c'(x) = i \\ x & \text{otherwise,} \end{cases}$$

is easily seen to be a retraction, and we can conclude that $G \in AR_n$. The additional properties can be verified using the z_i's.

(ii) → (i): We adjoin n new vertices $x_1, \ldots, x_n$ to G such that each x_i is connected to all $x \in V(G)$ with $c(x) \neq i$; the new graph is n-chromatic. The additional assumptions on G now guarantee that G is isometric in G', hence there is a retraction $r: G' \to G$. Taking $z_i := r(x_i)$, one has the desired result. □

3. The main results

We first have to introduce the following concept:

Definition 3.1. Let G be a graph, v a vertex of G. We say that v is *embeddable* if there is another vertex w connected to at least all the neighbours of v; equivalently, $G \backslash v$ is a retract of G.

It follows from 2.1 that if v is an embeddable vertex of G, then $G \in AR_n$ implies $G \backslash v \in AR_n$. The graphs D_4 and D_5 shown in Fig. 1 witness that $G \in AR_n$ cannot be concluded from $G \backslash v \in AR_n$; the recursion we are aiming at essentially says that $G \in AR_n$ follows if $G \backslash v \in AR_n$ holds for any extreme vertex v. (v is called *extreme* if for some vertex w, the distance $d_G(v, w)$ equals diam G.)

One more technical result has to be proved first.

Lemma 3.2. *Let $n \geq 3$, $G \in AR_n$, and $v, w \in V(G)$ with $l := d_G(v, w) \geq 3$. If v is not embeddable, then there is a neighbour v' of v with $d_G(v', w) = d_G(v, w) + 1$.*

Proof. We extend G by adding vertices $x_2, \ldots, x_l$ such that
(i) x_2 is connected to all neighbours of v, and
(ii) there are edges $x_2x_3, \ldots, x_{l-1}x_l, x_lw$.
The resulting graph G' is n-chromatic: given a colouring c of G, it is easy to extend by $c(x_2) := c(v)$ and then colour the path $x_3, \ldots, x_l$ (since $n \geq 3$).

G is not isometric in G', because otherwise, there would be a retraction $r: G' \to G$, and v would be embeddable into $r(x_2) \neq v$. But this means that for some neighbour v' of v, $d_G(v', w) > d_{G'}(v', w) = l$, which was to be shown. □

Remark. In the above Lemma, if $n = 2$, one can make a similar proof introducing $l - 2$ extra vertices instead of $l - 1$; in this case, the existence of v' comes from the fact that $d_G(v', w) \neq d_G(v, w)$ for all neighbours v' of v.

An important consequence of this lemma (and the remark) is:

Corollary 3.3. *If $G \in AR_n$ and* diam $G \geq 3$, *then each extreme vertex v of G is embeddable.*

This Corollary implies that a graph from AR_n can be dismantled by removing embeddable vertices, one at a time, until the remaining graph satisfies the conditions of 2.2. With the help of 1.2, one now gets:

Corollary 3.4. *For each $G \in AR_n$, there is a unique n-colouring of G, i.e., a unique homomorphism of G onto K_n (up to an automorphism of K_n).*

For the next theorem, we first introduce a seemingly smaller class than AR_n. We say that $G \in CAR_n$ if and only if
- G is n-chromatic, and
- whenever G is an isometric subgraph of an n-chromatic graph G' coloured by c', then there is a *colour-preserving* retraction from G' onto G.

Theorem 3.5. *Let G be n-chromatic. If* diam $G \geq 3$, *and if each extreme vertex $v \in V(G)$ is embeddable with $G \backslash v \in CAR_n$, then $G \in CAR_n$.*

Proof. Let G be isometric in G', and let c' be an n-colouring of G'. Let $v, w \in V(G)$ with $d_G(v, w) = \operatorname{diam} G$.

Since $G \backslash w$, too, is isometric in G', there is a colour-preserving retraction $r_w: G' \to G \backslash w$. We claim that r_w can be so chosen that for all $a \in V(G') \backslash V(G)$ satisfying $av \in E(G')$, $d_G(r_w(a), w) < d_G(v, w)$ holds.

Assume that $d_G(r_w(a), w) = d_G(v, w)$ for one such a. $a_v := r_w(a)$ is embeddable in some $\tilde{a}_v$ by the assumption of our theorem. By repetition of the same argument and diam $G \geqslant 3$, we finally have to get an $\tilde{a}_v$ into which a_v is embeddable and which satisfies $d_G(\tilde{a}_v, w) < d_G(v, w)$. It is clear that the vertex a can be mapped to $\tilde{a}_v$ instead of a_v to get a 'new' retraction r_w. We have to observe that this operation keeps r_w colour-preserving: this is an easy consequence of $G\setminus w \in CAR_n$ and 1.2. Applying that argument to all the $a \in V(G')$ in question gives us the extra property of r_w.

Let the graph G'' be obtained by identifying in G' each vertex $a \in V(G')\setminus V(G)$ satisfying $av \in E(G')$ with its image $r_w(a)$, keeping all the edges (i.e., $r_w(a)b \in E(G'')$ if $ab \in E(G')$) without producing double edges; since r_w is colour-preserving, the old colouring c' shows G'' is n-chromatic.

Because of the extra condition satisfied by r_w, G is still isometric in G'', and so is $G\setminus v$. Hence, we know there is a colour-preserving retraction r_v: $G'' \to G\setminus v$. If we define r': $G' \to G$ by

$$r'(x) := \begin{cases} v & \text{if } x = v \\ r_v(x) & \text{if } x \neq v,\ vx \notin E(G')\setminus E(G) \\ r_w(x) & \text{if } x \neq v,\ vx \in E(G')\setminus E(G), \end{cases}$$

it is easy to see that r' retracts G' in a colour-preserving way onto G. □

Corollary 3.6. $AR_n = CAR_n$.

Proof. Let $G \in AR_n$ with diam $G \geqslant 3$. (The case that diam $G = 2$ is handled by 2.2.) By induction hypothesis, $G\setminus v \in CAR_n$ for each extreme vertex $v \in V(G)$, and the above theorem yields $G \in CAR_n$. □

Corollary 3.7. *Let G be n-chromatic with* diam $G \geqslant 3$. *Then $G \in AR_n$ if and only if for each extreme $v \in V(G)$, v is embeddable and $G\setminus v \in AR_n$.*

Applying 3.7, it is now easy to see that the graph D_4 shown in Fig. 3(a) is in AR_3: v and w as indicated in the diagram are the only extreme vertices, both embeddable, and D_3 (see Fig. 3(b)) is in AR_3 by 2.2. Also, one can now see that $D_5 \notin AR_3$ (see Fig. 3(c)): w is extreme, but not embeddable.

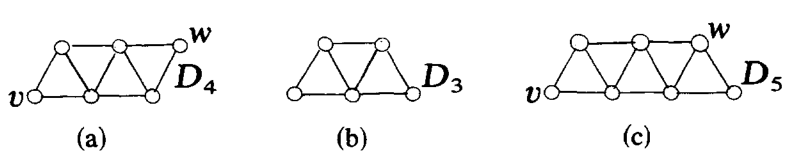


Fig. 3

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